The Mathematics Of Stock Option Valuation - Part Two
Solution Via Partial Differential Equations

Gary Schurman, MBE, CFA

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In Part One we explained why valuing a call option as a stand-alone asset using risk-adjusted discount rates will almost always lead to an incorrect value because the value determined in this manner will most likely be subject to arbitrage. If valuing the call as a stand-alone asset is an incorrect methodology then what methodology is correct? We need to calculate a call price that prohibits arbitrage. In this section we will use partial differential equations (also known as PDEs) to accomplish this task.

We will be modeling asset prices in discrete time. The asset pricing models that we employ will not be continuous in time intervals less than one year and therefore the derivatives of those asset price equations are not valid unless the change in time equals one year ($\delta t = 1$). Our purpose in this section is to demonstrate the mathematics of option valuation and therefore the assumption is that we fund the hedge portfolio at time $t = 0$ and unwind the hedge portfolio at time $t = 1$ with no intervening trades.

The One Period Economy (From Part One)

Our economy has two states of the world at time $t = 1$. In state w_u the stock price moves up to \$180 and in state w_d the stock price moves down to \$80. We have a call option on this stock with an exercise price of \$120 that can be exercised at period $t = 1$. If the stock price is above the exercise price the option will be exercised, otherwise it will be allowed to expire unexercised. The table below presents our one period economy and the two states of the world at time $t = 1$...

Table 1: The One Period Economy

State of the world	w_u	w_d
Stock price	\$180	\$80
Call price	\$60	\$0
Risk-free rate	0.05	0.05
Probability	0.50	0.50

We currently sit at $t = 0$ where the state-of-the-world at $t = 1$ is unknown. In Part One we estimated the stock price at time $t = 0$ to be \$100. In Part Two we will calculate the no-arbitrage value of the call at $t = 0$.

Legend of Symbols

B_t	=	Zero-coupon bond price at the end of time t
C_t	=	Call option price at the end of time t
P_t	=	Hedge portfolio value at the end of time t
S_t	=	Stock price at the end of time t
S_u	=	Stock price at time $t = 1$ given that random variable $z = +1$
S_d	=	Stock price at time $t = 1$ given that random variable $z = -1$
θ_1	=	Number of shares of stock in the hedge portfolio
θ_2	=	Number of call options in the hedge portfolio
r	=	Risk-free rate of interest
t	=	Time period in years
z	=	Random variable that takes the value of +1 or -1 at $t = 1$

A Discrete Time Model For Stock Price

We will model stock price at time t as a linear function of stock price at time zero, drift and an innovation. The drift term will be deterministic in that its value at time t is known at time zero. The innovation term will be random in that its value at time t is unknown at time zero. The equation for stock price at time t as a function of the variables t and z is...

$$S_t = S_0 + \mu t + \sigma z \tag{1}$$

For our one period economy the variable μ equals 30 and the variable σ equals 50. The equation for stock price at time $t = 1$ for our one period economy using equation (1) as a template is...

$$S_1 = 100 + 30t + 50z \tag{2}$$

A partial derivative is the derivative of a function of two or more variables with respect to a single variable while the other variables are considered to be constant. The partial deriviatives of stock price equation (1) with respect to the variables t and z are...

$$\frac{\delta S_t}{\delta t} = \mu \ \ and \ \ \frac{\delta S_t}{\delta z} = \sigma \tag{3}$$

The total change in stock price between time t and time $t + \delta t$ is the sum of the partial derivatives. Note that in this case the second order and higher partial derivatives are zero. The equation for the change in stock price at time t is...

$$\delta S_t = \frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta z}\delta z \tag{4}$$

As an example, stock price in our one period economy decreases from \$100 to \$80 at $t = 1$ in state of the world w_d. In this case the deterministic variable t increases from zero to one ($\delta t = 1$) and the random variable z decreases from zero to negative one ($\delta z = -1$). The change in stock price using the partial derivatives in equation (4) is...

$$\begin{aligned}\delta S_0 &= \mu\,\delta t + \sigma\,\delta z \\ &= (30)(1) + (50)(-1) \\ &= -20\end{aligned} \tag{5}$$

A Discrete Time Model For Call Price

Our task is to derive an equation for call price at $t = 0$ and although we do not know the exact equation we do know its general form. We will model call price as a function of time (t) and stock price (S_t). The general form of the equation for call price is...

$$C_t = C(S_t, t) \tag{6}$$

Call price is a function of the underlying stock price for obvious reasons. In cases where the stock price exceeds the call exercise price at expiration the payoff on the call increases as stock price increases. Call price is also a function of time. Given the time value of money the closer the call is to expiration the more valuable is the call. Note that this is the case because the call payoffs at $t = 1$ are fixed and will not hold in the continuous time model.

The total change in stock price between time t and time $t + \delta t$ is the sum of the partial derivatives. The equation for the total change in call price at time t is...

$$\delta C_t = \frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\delta S_t \tag{7}$$

Pricing Derivatives Via PDEs

Since both the stock and the call option on that stock are driven by the same random process (i.e. the random variable z) these two assets can be combined in one portfolio such that the randomness of one asset offsets the randomness of the other resulting a portfolio that is risk-free. The price of the call option will be determined via the following steps...

1 Create the hedge portfolio and derive the PDE
2 Find the equation that solves the PDE derived in step 1
3 Determine the parameters of the equation in step 2 that are consistent with the boundary conditions
4 Use the equation in step 3 to determine call price at $t = 0$

We will follow these steps to price the call option in our one period economy.

Step One - Create The Hedge Portfolio And Derive The PDE

We will create a hedge portfolio that consists of θ_2 call options and θ_1 shares of the underlying stock. The equation for portfolio value at time t is...

$$P_t = \theta_1 S_t + \theta_2 C_t \tag{8}$$

The equation for the change in portfolio value at time t is...

$$\delta P_t = \theta_1 \delta S_t + \theta_2 \delta C_t \tag{9}$$

Note that change in stock price equation (4) can be substituted for δS_t and change in call price equation (7) can be substituted for δC_t in equation (9) above. After making these substitutions the equation for change in portfolio value at time t becomes...

$$\delta P_t = \theta_1 \left[\frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta z}\delta z\right] + \theta_2 \left[\frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\delta S_t\right] \tag{10}$$

We will now substitute equation (4) for the δS_t in equation (10) above. After making this substitution the equation for change in portfolio value at time t becomes...

$$\begin{aligned}\delta P_t &= \theta_1 \left[\frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta z}\delta z\right] + \theta_2 \left[\frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\left\{\frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta z}\delta z\right\}\right] \\ &= \theta_1 \left[\frac{\delta S_t}{\delta t}\delta t + \frac{\delta S_t}{\delta z}\delta z\right] + \theta_2 \left[\frac{\delta C_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta z}\delta z\right]\end{aligned} \tag{11}$$

Note that the change in portfolio value is a function of the change in t, which is deterministic, and the change in z, which is random. The portfolio is not risk-free in that we do not know the value of the random variable z at time $t+\delta t$ at time t. If we can remove the randomness from the portfolio then the portfolio value at time $t+\delta t$ is known at time t and therefore is risk-free. Because we set the portfolio weights θ_1 and θ_2 at time t we can remove the random variable z by making the following definitions...

$$\theta_1 = \frac{\delta C_t}{\delta S_t} \;\; and \;\; \theta_2 = -1 \tag{12}$$

After making these substitutions the equation for change in portfolio value at time t becomes...

$$\begin{aligned}\delta P_t &= \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta t}\delta t + \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta z}\delta z - \frac{\delta C_t}{\delta t}\delta t - \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta t}\delta t - \frac{\delta C_t}{\delta S_t}\frac{\delta S_t}{\delta z}\delta z \\ &= -\frac{\delta C_t}{\delta t}\delta t\end{aligned} \tag{13}$$

Because the change in portfolio value is known at time t the portfolio is risk-free and should earn the risk-free rate. The equation for the change in portfolio value at time t becomes...

$$rP_t\delta t = -\frac{\delta C_t}{\delta t}\delta t \tag{14}$$

We now substitute the P_t in equation (14) above with the definition of portfolio value equation (8). The equation for the change in portfolio value at time t becomes...

$$\begin{aligned}r\Big[\theta_1 S_t + \theta_2 C_t\Big]\delta t &= -\frac{\delta C_t}{\delta t}\delta t \\ r\theta_1 S_t \delta t + r\theta_2 C_t \delta t + \frac{\delta C_t}{\delta t}\delta t &= 0\end{aligned} \tag{15}$$

We now substitute the definitions in equation (12) for θ_1 and θ_2 in equation (15) above. The equation for the change in portfolio value at time t becomes...

$$r\frac{\delta C_t}{\delta S_t}S_t\delta t + \frac{\delta C_t}{\delta t}\delta t - rC_t\delta t = 0 \tag{16}$$

Since the δt terms are common to all factors they can be eliminated from equation (16) to obtain the partial differential equation (PDE) below...

$$r\frac{\delta C_t}{\delta S_t}S_t + \frac{\delta C_t}{\delta t} - rC_t = 0 \tag{17}$$

Step Two - Find A Solution To The PDE

The solution to a partial differential equation is another equation. We need an equation such that when we take it's derivatives and drop them into the PDE equation (17) it equals zero. Our first guess as to a solution to the PDE is the following equation...

$$C_t = NS_t - B_0 e^{rt} \tag{18}$$

The equation above replicates a call option by buying N shares of the underlying stock (a long position) and selling a risk-free bond (a short position). To see if this equation solves PDE equation (17) we must first take the derivatives of equation (18) which are...

$$\frac{\delta C_t}{\delta S_t} = N \;\; and \;\; \frac{\delta C_t}{\delta t} = -rB_0 e^{rt} \tag{19}$$

We then drop the solution equation derivatives into PDE equation (17) and see if it equals zero (which it does)...

$$\begin{aligned} r\frac{\delta C_t}{\delta S_t} S_t + \frac{\delta C_t}{\delta t} - rC_t &= 0 \\ rNS_t - rB_0 e^{rt} - r\Bigg[NS_t - B_0 e^{rt}\Bigg] &= 0 \\ rNS_t - rB_0 e^{rt} - rNS_t + rB_0 e^{rt} &= 0 \end{aligned} \tag{20}$$

Conclusion: Equation (18) is a solution to the PDE.

Step Three - Determine The Solution Equation Parameters

We have proved that equation (18) is a solution to the PDE equation (17). In order to use that equation to calculate call price at $t = 0$ we have to solve for the parameters N and B_0. We know that the solution equation must equal the call payoffs at $t = 1$. The call payoffs at $t = 1$ are our boundary conditions. Our two boundary conditions are...

$$NS_u - B_0 e^{rt} = C_u \tag{21}$$

$$NS_d - B_0 e^{rt} = C_d \tag{22}$$

We will first subtract equation (22) from equation (21) to eliminate the variable $B_0 e^{rt}$ which leaves us the variable N to solve for. The solution to equation parameter N is...

$$\begin{aligned} N(S_u - S_d) &= C_u - C_d \\ N &= \frac{C_u - C_d}{S_u - S_d} \\ N &= \frac{60 - 0}{180 - 80} \\ N &= 0.60 \end{aligned} \tag{23}$$

Now that we solved for parameter N we can solve for parameter B_0. We will add equation (21) and equation (22) and solve for B_0 as follows...

$$\begin{aligned} N(S_u + S_d) - 2B_0 e^{rt} &= C_u + C_d \\ N(S_u + S_d) - (C_u + C_d) &= 2B_0 e^{rt} \\ \Bigg[N\Bigg\{\frac{S_u + S_d}{2}\Bigg\} - \Bigg\{\frac{C_u + C_d}{2}\Bigg\}\Bigg]e^{-rt} &= B_0 \\ \Bigg[(0.60)\Bigg\{\frac{180 + 80}{2}\Bigg\} - \Bigg\{\frac{60 + 0}{2}\Bigg\}\Bigg]e^{-0.05} &= 45.66 \end{aligned} \tag{24}$$

Step 4 - Determine Call Price At Time Zero

The equation for call price at $t = 0$ using equation (17) as a template is...

$$\begin{aligned} C_0 &= NS_0 - B_0 e^{r \times 0} \\ &= NS_0 - B_0 \end{aligned} \tag{25}$$

We can replace the parameters N and B_0 in equation (25) above with equation (23) and (24), respectively. The equation for call price after making these substitutions is...

$$\begin{aligned} C_0 &= (0.60)(100) - 45.66 \\ &= 14.34 \end{aligned} \tag{26}$$

Proof

We will now determine if the call price of \$14.34 prevents arbitrage. To refresh our memory, the requirements for a portfolio that prevents arbitrage are...

1 Has zero cost to set up
2 Has non-negative values in the future
3 Has non-positive values in the future

The cost to set up the hedge portfolio is...

Proceeds from the sale of one call	14.34
Proceeds from shorting a risk-free bond	45.66
Purchase 0.60 shares of stock	-60.00
Net cost to set up at $t = 0$	0.00

The value of the portfolio at time $t = 1$ is...

State of the world	w_u	w_d
Sell 0.60 shares of stock	108.00	48.00
Payoff short position in one call	-60.00	0.00
Payoff short position in bond	-48.00	-48.00
Net portfolio value at $t = 1$	0.00	0.00

Conclusion: A call option price of \$14.34 prevents arbitrage and is therefore the valid call price at $t = 0$.